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ON REGULAR IMPLEMENTABILITY USING CONTROLLERS WITH A PRIORI INPUT/OUTPUT STRUCTURE

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Abstract— **In this paper we deal with the problem of finding necessary and sufficient conditions for a behavior to be regularly implementable using controllers in which an a priori given part of the control variables is free or maximally free. We will solve the above problems in both the full and the partial interconnection case.**

I. INTRODUCTION

An important issue in the behavioral approach to control is implementability. Implementability deals with the question which system behaviors can be achieved by interconnecting a given system with a controller. In the behavioral framework this is made precise as follows. Given is a plant behavior with two types of variables, the variable w to be controlled and the control variable c on which we are allowed to put restrictions by interconnecting these variables with a controller. In the behavioral approach we treat a controller as an additional system behavior, called controller behavior. The space of all trajectories w possible after interconnecting the plant behavior with the controller behavior is called the manifest controlled behavior. A behavior is called implementable if it is possible to obtain it as manifest controlled behavior in this way. In the context of pole placement and stabilization an important role is played by regular interconnection. A given behavior is called regularly implementable if it can be achieved by a controller behavior that does not impose restrictions on the control variable that are already present in the plant, equivalently, the number of outputs of the associated full controlled behavior is equal to the sum of the number of outputs of the plant and the number of outputs of the controller. In [3], for a given plant behavior a characterization was given of all implementable behaviors and in [4] a characterization was given of all regularly implementable behaviors.

In this paper we deal with the problems of finding necessary and sufficient conditions for a behavior to be regularly implementable using a controller in which an a priori given part of the control variables is free or maximally free. In other words, we will require a priori given components of the control variable to be part of controller input, or even to be controller input. The remaining part of the control variable then necessarily contain, the controller output, or is equal to the controller output. This problem was studied before in [8], but only partial results were obtained. Here, we resolve these problems in both the full and the partial interconnection case.

II. LINEAR DIFFERENTIAL SYSTEMS

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^q$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ (the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a real polynomial matrix R with q columns such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0)\}$. Any such dynamical system Σ is called a linear differential system. The set of all linear differential systems with q variables is denoted by $\mathcal{L}^q$. Since the behavior $\mathfrak{B}$ of the system Σ is the central item, we will mostly speak about the system $\mathfrak{B} \in \mathcal{L}^q$ (instead of $\Sigma \in \mathcal{L}^q$). Henceforth, in this paper we will suppress the notation '$\frac{d}{dt}$', and write Rw instead of $R(\frac{d}{dt})w$.

The behavioral approach makes a distinction between the behavior as the space of all solutions to a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a representation of the behavior. If a behavior $\mathfrak{B}$ is represented by $Rw = 0$ then we call this a kernel representation of $\mathfrak{B}$, and we often write $\mathfrak{B} = \ker(R)$.

Suppose R has p rows. Then the kernel representation is said to be minimal if every other kernel representation of $\mathfrak{B}$ has at least p rows. A given kernel representation $\mathfrak{B} = \ker(R)$ is minimal if and only if the polynomial matrix R has full row rank (see [10], theorem 3.6.4). The number of rows in any minimal kernel representation of $\mathfrak{B}$ is denoted by $p(\mathfrak{B})$. This number is called the output cardinality of $\mathfrak{B}$. It corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. The number of remaining components is called the input cardinality of $\mathfrak{B}$ and is denoted by $m(\mathfrak{B})$. Thus $m(\mathfrak{B}) = q - p(\mathfrak{B})$.

We now review some facts on elimination. Again let $\mathfrak{B} \in \mathcal{L}^q$ with system variable $w = (w_1, w_2)$. Let P_{w_1} denote the projection onto the w_1-component. Then the set $P_{w_1}\mathfrak{B}$ of all w_1 for which there exists w_2 such that $(w_1, w_2) \in \mathfrak{B}$ is again a linear differential system. In this paper we denote $P_{w_1}\mathfrak{B}$ by $\mathfrak{B}_{w_1}$. We call $\mathfrak{B}_{w_1}$ the system obtained by eliminating w_2 from $\mathfrak{B}$ (see [10] section 6.2.2). If $\mathfrak{B} = \ker\begin{pmatrix} R_1 & R_2 \end{pmatrix}$, then a representation for $\mathfrak{B}_{w_1}$ is obtained as follows: choose a unimodular matrix U such that $UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix}$, with R_{12} full row rank, and conformably partition $UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}$. Then $\mathfrak{B}_{w_1} = \ker(R_{21})$ (see [10], section 6.2.2).

We now define the notion of free and maximally free variables (see [9] section 2.9).

Definition 1: Let $\mathfrak{B} \in \mathcal{L}^{q_1+q_2}$ with manifest variable (w_1, w_2). We will call w_2 free in $\mathfrak{B}$ if for any choice of $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ there exists w_1 such that $(w_1, w_2) \in \mathfrak{B}$. We call w_2 maximally free if it is free, and we can not enlarge this set with components from w_1 and still continue to have freeness for this enlarged set of variables.
It turns out that in general a behavior $\mathfrak{B}$ has many maximally free sets of variables. However, the number of components of every maximally free set of variables is the same, and is equal to $m(\mathfrak{B})$, the input cardinality of $\mathfrak{B}$.

Definition 2: We say a behavior $\mathfrak{B}$ is autonomous if $m(\mathfrak{B}) = 0$.
If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is autonomous if and only if R has full column rank.

III. Review of implementability and problem formulation

In this section we will first briefly recall the notions of implementability and regular implementability. Next, we will introduce the problems studied in this paper.

A. Review of implementability

Definition 3: Let $\mathcal{P} \in \mathcal{L}^q$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathcal{L}^q$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. This controlled behavior is also an element of $\mathcal{L}^q$. The full interconnection is called regular if $p(\mathcal{P} \cap \mathcal{C}) = p(\mathcal{P}) + p(\mathcal{C})$.

From [5] if $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ then $\mathcal{K}$ is implementable with respect to $\mathcal{P}$ by full interconnection if and only if there exists a polynomial matrix F such that $R = FK$. The property of regular implementability turns out to be equivalent with the existence of such polynomial matrix F with, in addition, $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$ (see [5], theorem 9).

Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^q$ and c in $\mathbb{R}^k$. The variable w should be interpreted as the variable to be controlled, the variable c are those through which we can interconnect the plant with a controller, and is called the control variable. Let $\mathcal{C} \in \mathcal{L}^k$ be a controller behavior, with variable c.

Definition 4: The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is defined as the system behavior $\mathcal{K}_{\text{full}}(\mathcal{C}) \in \mathcal{L}^{q+k}$, given by $\mathcal{K}_{\text{full}}(\mathcal{C}) = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}$. This behavior is called the full controlled behavior, and is denoted by $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$.

Definition 5: The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is called regular if the output cardinality of the full controlled behavior is the sum of the output cardinalities of the plant and the controller, i.e., $p(\mathcal{K}_{\text{full}}(\mathcal{C})) = p(\mathcal{P}_{\text{full}}) + p(\mathcal{C})$.

This condition is equivalent to: $\mathcal{C}$ does not re-impose restriction on $\mathcal{K}_{\text{full}}(\mathcal{C})$ that are already present in $\mathcal{P}_{\text{full}}$ (see [6] and [7]).

Definition 6: The behavior $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w \in \mathcal{L}^q$ that is obtained by eliminating c from $\mathcal{K}_{\text{full}}(\mathcal{C})$ is called the manifest controlled behavior.

Definition 7: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$. The hidden behavior $\mathcal{N} \in \mathcal{L}^q$ is the behavior consisting of the to-be-controlled variable trajectories that can occur if the control variables are restricted to be equal to zero:

$$\mathcal{N} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \mid (w, 0) \in \mathcal{P}_{\text{full}}\}.$$

Let $\mathcal{K} \in \mathcal{L}^q$ be a given behavior, which should be interpreted as a 'desired' behavior. A fundamental question is whether this $\mathcal{K}$ can be achieved as controlled behavior:

Definition 8: If there exists $\mathcal{C} \in \mathcal{L}^k$ such that $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ then $\mathcal{K}$ is called implementable by partial interconnection (through c, with respect to $\mathcal{P}_{\text{full}}$). If there exists a $\mathcal{C} \in \mathcal{L}^k$ such that $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ and $p(\mathcal{K}_{\text{full}}(\mathcal{C})) = p(\mathcal{P}_{\text{full}}) + p(\mathcal{C})$, then we call $\mathcal{K}$ regularly implementable by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$).
Necessary and sufficient condition for implementability and regular implementability by partial interconnection were obtained in [3] and [4]. We review these conditions here for quick reference:

Proposition 9:
1) $\mathcal{K} \in \mathcal{L}^q$ is implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$.
2) $\mathcal{K} \in \mathcal{L}^q$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$ and $\mathcal{K}$ is regularly implementable with respect to $(\mathcal{P}_{\text{full}})_w$ by full interconnection.

B. Problem formulation

As mentioned above, necessary and sufficient conditions for regular implementability are obtained in [3] and [4]. In these papers the authors deal with controllers on which there is no input/output constraint. Often, a controller uses information on the plant measurements, and this set of measured variables is not allowed to be constrained by the controller. In other words, it is a naturally emerging constraint that a given part of the control variables *is free in the controller*. The problem of regular implementability using controllers in which an a priori given part of the control variables is free was also considered in [8]. Consider the following example from [8]:

Example 10: Consider a single tank system as shown in figure 1. On top of the tank there is an inlet from which variable flow of water $u(t)$ can get in to the tank. There is an opening at the bottom of the tank connected to a pump through which we can pump in/out the water from tank. The flow which is pumped out of the tank is denoted by $y(t)$.

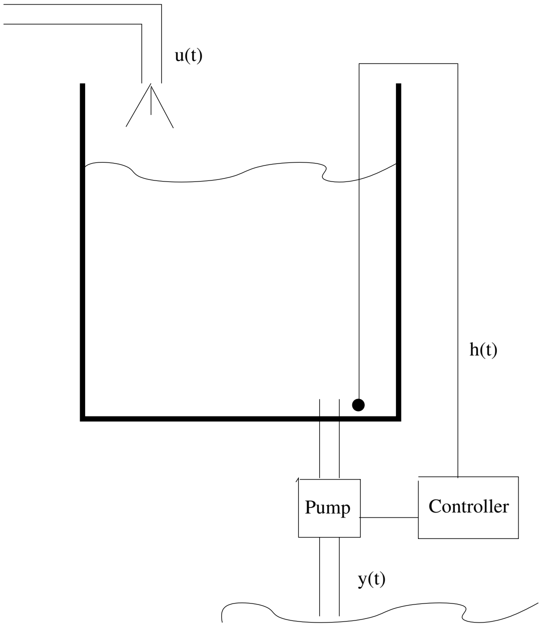

Fig. 1. Single tank system

The tank is also equipped with a sensor which measures the change in volume inside the tank, the measurement of the sensor is denoted by $h(t)$. The mathematical model of the plant is given by

$$h(t) = u(t) - y(t). \tag{1}$$

Consider the following control problem. Given $h(t), u(t)$ as control variables we want to design a controller which will keep the level inside the tank constant, i.e $y(t) = u(t)$. The problem is mathematically formulated as follows:

Given are $\mathcal{P}_{\text{full}} = \{(u, y, h)) \mid -u(t) + y(t) + h(t) = 0\}$ with plant variable (w, c) where $w = (u(t), y(t)), c = (y(t), h(t))$ and $\mathcal{K} = \{(u(t), y(t))) \mid -u(t) + y(t) = 0\}$. From proposition 9 one can check that this $\mathcal{K}$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$, and a controller which accomplishes this task is given by $h(t) = 0$. Here the variable $h(t)$ is the measurement coming from the system sensor. From physical intuition this controller is not realizable, as restricting sensor measurement practically does not make sense. Therefore, even though given $\mathcal{K}$ is regularly implementable it is not practically realizable.

From this example it is evident that, in this kind of situations where we have constraints on some part of the control variables, all regularly implementable behaviors are not practically realizable. In [8] preliminary results for a behavior to be regularly implementable using controllers in which an a priori given part of the control variables is free are obtained. In the present paper we will establish necessary and sufficient conditions for the existence of such controllers in terms of $\mathcal{P}_{\text{full}}$, $\mathcal{K}$, and the partition of the control variables.

IV. Regular implementability using controllers with a priori input/output structure

In this section we look at the problem of finding necessary and sufficient conditions for a behavior to be regularly implementable using controllers in which some part of the variables is free. In addition we look at the problem of regular implementability by controllers in which some part of the variables is maximally free. We solve these problems in both the full and the partial interconnection case.

A. Full interconnection

Let $\mathcal{P}$, $\mathcal{K} \in \mathcal{L}^{q_1+q_2}$ with plant variable (w_1, w_2). We will consider controllers $\mathcal{C} \in \mathcal{L}^{q_1+q_2}$ with control variable (w_1, w_2). We first look at the problem of finding conditions on the behavior $\mathcal{K}$ to be regularly implementable by a controller $\mathcal{C}$ in which w_2 is free. Apart from the condition that $\mathcal{K}$ should be regularly implementable, an additional condition plays a role. This is stated in the following theorem:

Theorem 11: Let $\mathcal{P}$, $\mathcal{K} \in \mathcal{L}^{q_1+q_2}$ with plant variable (w_1, w_2). Then $\mathcal{K}$ is regularly implementable through full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is free if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}$,
2) $p((\mathcal{K})_{w_2}) \leq p(\mathcal{P})$.

Before proving the theorem we will establish some results which are useful in the proof.

Let $\mathcal{K} \in \mathcal{L}^{q_1+q_2}$ with plant variable (w_1, w_2), and define $\mathcal{N}(\mathcal{K}) = \{w_1 \mid (w_1, 0) \in \mathcal{K}\}$. Then we have the following lemma

Lemma 12: Let $\mathcal{K} \in \mathcal{L}^{q_1+q_2}$ with system variable (w_1, w_2). Then

$$p(\mathcal{N}(\mathcal{K})) = p(\mathcal{K}) - p((\mathcal{K})_{w_2}).$$

Proof: Let $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ give a minimal kernel representation of $\mathcal{K}$. Then there exists a unimodular matrix U such that $UK = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}$ and K_{11}, K_{22} have full row rank. Therefore $\mathcal{N}(\mathcal{K}) = \ker(K_{11})$, so $p(\mathcal{N}(\mathcal{K})) = \text{rank}(K_{11})$. Since $\text{rank}(K_{11}) = \text{rank}(K) - \text{rank}(K_{22}) = p(\mathcal{K}) - p((\mathcal{K})_{w_2})$, we obtain $p(\mathcal{N}(\mathcal{K})) = p(\mathcal{K}) - p((\mathcal{K})_{w_2})$. ■

We also use an important result obtained as lemma 4.73 in [8] to prove the theorem. This result is stated as a lemma here.

Lemma 13: Let C and M be polynomial matrices with the same number of columns. There exists a polynomial matrix V such that $C + VM$ has full row rank if and only if

$$\text{rank}\begin{pmatrix} M \\ C \end{pmatrix} \geq \text{rowdim}(C).$$

Using the above lemmas we prove theorem 11.

Proof of theorem 11: (if) Let $R = \begin{pmatrix} R_1 & R_2 \end{pmatrix}$ and $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ give minimal kernel representations of the behaviors $\mathcal{P}$ and $\mathcal{K}$, respectively.

From condition 1, there exists a F such that $R = FK$ and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Take W such that $\begin{pmatrix} F \\ W \end{pmatrix}$ forms a unimodular matrix. From [5] theorem

11, WK have full row rank, $\ker(WK)$ regularly implements $\mathcal{K}$, and a parameterization of all controllers which regularly implement $\mathcal{K}$ is given by $GR + UWK$, where G is an arbitrary polynomial matrix, and U is unimodular.

From the above arguments, $\ker\begin{pmatrix} R_1 & R_2 \\ WK_1 & WK_2 \end{pmatrix} = \ker\begin{pmatrix} K_1 & K_2 \end{pmatrix}$, which implies $\mathcal{N}(\mathcal{K})=\ker\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix}$. Therefore,

$$p(\mathcal{N}(\mathcal{K})) = \operatorname{rank}\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix}. \tag{2}$$

Since $\ker(WK)$ regularly implements $\mathcal{K}$, we have

$$\begin{aligned} p(\mathcal{K}) &= p(\mathcal{P}) + \operatorname{rank}(WK) \\ &= p(\mathcal{P}) + \operatorname{rowdim}(WK) \\ &= p(\mathcal{P}) + \operatorname{rowdim}(WK_1). \end{aligned}$$

This implies $p(\mathcal{P}) = p(\mathcal{K}) - \operatorname{rowdim}(WK_1)$.

Condition 2 together with lemma 12 imply that $p(\mathcal{K}) - p(\mathcal{N}(\mathcal{K})) \leq p(\mathcal{K}) - \operatorname{rowdim}(WK_1)$. This implies $p(\mathcal{N}(\mathcal{K})) \geq \operatorname{rowdim}(WK_1)$, so $\operatorname{rank}\begin{pmatrix} R_1 \\ WK_1 \end{pmatrix} \geq \operatorname{rowdim}(WK_1)$.

Using this last inequality, by lemma 13 there exists a G_0 such that $G_0R_1 + WK_1$ has full row rank. Define $\mathcal{C}_0 = \ker(G_0R + WK)$. Then $\mathcal{C}_0$ is a controller in which w_2 is free and that regularly implements $\mathcal{K}$.

(only if) Let $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ be a minimal kernel representation of a controller $\mathcal{C}$ in which w_2 is free and which regularly implements $\mathcal{K}$. We know that w_2 is free in $\mathcal{C}$ if and only if C_1 has full row rank.

As $\ker\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix} = \ker\begin{pmatrix} K_1 & K_2 \end{pmatrix}$, we have

$$\begin{aligned} p(\mathcal{N}(\mathcal{K})) &= \operatorname{rank}(K_1) \\ &= \operatorname{rank}\begin{pmatrix} R_1 \\ C_1 \end{pmatrix} \\ &\geq \operatorname{rank}(C_1) \\ &= \operatorname{rank}(K) - \operatorname{rank}(R) \\ &= p(\mathcal{K}) - p(\mathcal{P}). \end{aligned}$$

Therefore $p(\mathcal{N}(\mathcal{K})) \geq p(\mathcal{K}) - p(\mathcal{P})$, so by lemma 12 $p(\mathcal{K}) - p((\mathcal{K})_{w_2}) \geq p(\mathcal{K}) - p(\mathcal{P})$, which implies $p((\mathcal{K})_{w_2}) \leq p(\mathcal{P})$. ■

Condition 2 of the above theorem requires $p((\mathcal{K})_{w_2}) \leq p(\mathcal{P})$, equivalently $q_2 - m((\mathcal{K})_{w_2}) \leq (q_1 + q_2) - m(\mathcal{P})$, therefore $m((\mathcal{K})_{w_2}) \geq m(\mathcal{P}) - q_1$. In other words for $\mathcal{K}$ to be regularly implementable by a controller in which w_2 is free the input cardinality of the projected behavior $(\mathcal{K})_{w_2}$ should not be less than the difference between the input cardinality of $\mathcal{P}$ and q_1.

We now derive conditions on $\mathcal{K}$ to be regularly implementable by a controller $\mathcal{C}$ in which w_2 is *maximally free*. It is evident that for w_2 to be maximally free in $\mathcal{C}$ it should be free in $\mathcal{C}$. Therefore the class of controllers which regularly implement $\mathcal{K}$ and in which w_2 is maximally free forms a subset of the controllers which regularly implement $\mathcal{K}$ and in which w_2 is free. This fact is used in proving the following theorem which gives necessary and sufficient conditions for $\mathcal{K}$ to be regularly implementable by a controller in which w_2 is maximally free.

Theorem 14: Let $\mathcal{P}$, $\mathcal{K} \in \mathfrak{L}^{q_1+q_2}$ with plant variable (w_1, w_2). Then $\mathcal{K}$ is regularly implementable through full interconnection with respect to $\mathcal{P}$ using a controller $\mathcal{C}$ in which w_2 is maximally free if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable,
2) $p((K)_{w_2}) \leq p(\mathcal{P})$,
3) $p(\mathcal{K}) = q_1 + p(\mathcal{P})$, where q_1 is the size of w_1.

Proof: *(only if)* Let $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, $R = \begin{pmatrix} R_1 & R_2 \end{pmatrix}$ give minimal kernel representations of the behaviors $\mathcal{K}$ and $\mathcal{P}$, and let $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ be a minimal kernel representation of $\mathcal{C}$ which regularly implements $\mathcal{K}$ and in which w_2 is maximally free (w_2 is maximally free in $\mathcal{C}$ if and only if C_1 is square and nonsingular). This class of controllers is a subclass of the controllers treated in theorem 11, therefore condition 2 directly follows from theorem 11.

As $\ker\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix} = \ker\begin{pmatrix} K_1 & K_2 \end{pmatrix}$, we have

$$\begin{aligned} p(\mathcal{K}) &= \operatorname{rank}\begin{pmatrix} R_1 & R_2 \\ C_1 & C_2 \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} R_1 & R_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} C_1 & C_2 \end{pmatrix} \\ &= p(\mathcal{P}) + \operatorname{rowdim}(C_1) \\ &= p(\mathcal{P}) + \operatorname{coldim}(C_1) \\ &= p(\mathcal{P}) + q_1 \end{aligned}$$

(if) Let $\mathcal{K}= \ker\begin{pmatrix} K_1 & K_2 \end{pmatrix}$, $\mathcal{P}= \ker\begin{pmatrix} R_1 & R_2 \end{pmatrix}$ be minimal representations. Then from condition 1, regular implementability of $\mathcal{K}$, there exists a matrix F such that $\begin{pmatrix} FK_1 & FK_2 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \end{pmatrix}$, and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose W such that $\begin{pmatrix} F \\ W \end{pmatrix}$ forms a unimodular matrix. Then again by [5], theorem 11 a parameterization of all controllers which regularly implement $\mathcal{K}$ with respect to $\mathcal{P}$ is given by $\begin{pmatrix} C_1 & C_2 \end{pmatrix}=\begin{pmatrix} GR_1 + UWK_1 & GR_2 + UWK_2 \end{pmatrix}$, where G is an arbitrary polynomial matrix, and U is unimodular. Condition 2 implies that there exists a G such that $GR_1 + WK_1$ has full row rank.

From condition 3 we have $p(\mathcal{K}) - p(\mathcal{P}) = q_1$. This is equivalent to $\operatorname{rank}\begin{pmatrix} R_1 & R_2 \\ GR_1 + WK_1 & GR_2 + WK_2 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} R_1 & R_2 \end{pmatrix} = \operatorname{coldim}(GR_1 + WK_1)$, which in turn is equivalent to $\operatorname{rank}\begin{pmatrix} GR_1 + WK_1 & GR_2 + WK_2 \end{pmatrix} = \operatorname{coldim}(GR_1 + WK_1)$. Therefore $\operatorname{rowdim}(GR_1 + WK_1) = \operatorname{coldim}(GR_1 + WK_1)$ which implies that $GR_1 + WK_1$ is square.

Therefore condition 2 and 3 combinedly implies that there exists a matrix G such that $GR_1 + WK_1$ is square and nonsingular. Define $\mathcal{C} = \ker\begin{pmatrix} GR_1 + WK_1 & GR_2 + WK_2 \end{pmatrix}$. Then $\mathcal{C}$ is a controller in which w_2 is maximally free and which regularly implements $\mathcal{K}$ by full interconnection with respect to $\mathcal{P}$. ■

The following remarks relates the results obtained in this paper and the results obtained in [1].

Remark 1: Theorem 8 of [1] says that if $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ is a regular interconnection, then there always exists a partition of the variable w such that the interconnection can be viewed as a feedback interconnection. Feedback interconnection imposes an input output structure on the plant and the controller. The resulting input output partition of w in a controller may not coincide with the desired input output partition which is given a priori.

In this paper we precisely deal with the possibility of attaining the prespecified input output partition of the variable w in a controller which regularly implements $\mathcal{K}$.

Remark 2: From theorem 9 of [1] if w_2 is a part of the output of the plant, there always exists a controller in which w_2 is free and regularly implement $\mathcal{K}$ by full interconnection. This is also evident from our theorem 11, as the assumption of w_2 to be a part of the output of the plant forces the $\mathcal{K}$ to satisfy the sufficient conditions given in theorem 11.

Remark 3: One can show that if we assume w_2 to be part of the output of the plant, then all regularly implementable behaviors through full interconnection by controllers in which w_2 is maximally free are necessarily autonomous. This forces the desired behavior $\mathcal{K}$ to satisfy the sufficient conditions mentioned in theorem 14, hence guarantees the existence of a controller which regularly implements $\mathcal{K}$ and in which w_2 is maximally free.

From the above remarks it is evident that the results obtained in this paper are more general than the results given in [1].

B. Partial interconnection

We now deal with the problem of finding necessary and sufficient conditions for a behavior $\mathcal{K}$ to be regularly implementable by partial interconnection, using a controller in which an a priori given part of the control variables is free. We will solve this problem by reducing it to the full interconnection case.

In the sequel, the interconnected behavior $\mathcal{L}_{\text{full}}(\mathcal{K}) = \mathcal{P}_{\text{full}} \wedge_w \mathcal{K}$ plays an important role, which is summarized in the following proposition obtained in [6] (see also [7]).

Proposition 15: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ with system variable (w, c). Then $\mathcal{K} \in \mathcal{L}^q$ is regularly implementable by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$ if and only if the following two conditions hold:

1) $\mathcal{K}$ is implementable,
2) $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ is regularly implementable by full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$.

From [6] and [7], if $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection, then $\mathcal{C}' = \mathcal{C} + \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ with respect to $(\mathcal{P}_{\text{full}})_c$, and $\mathcal{C}'$ also regularly implements $\mathcal{K}$ by partial interconnection through c with respect to $\mathcal{P}_{\text{full}}$.

We now prove the following theorem, which gives necessary and sufficient conditions on $\mathcal{K}$ to be regularly implementable through partial interconnection by a controller $\mathcal{C}$ in which part of the control variables is free. In the following we consider a full plant behavior $\mathcal{P}_{\text{full}}$ with system variable (w, c). We assume c to be partitioned as $c = (c_1, c_2)$, and we will require c_2 to be free in the controllers that we are allowed to use.

Theorem 16: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ with system variable (w, c). Partition $c = (c_1, c_2)$. Then $\mathcal{K} \in \mathcal{L}^q$ is regularly implementable through c with respect to $\mathcal{P}_{\text{full}}$ using a controller in which c_2 is free if and only if the following conditions hold:

1) $\mathcal{K}$ is regularly implementable through c with respect to $\mathcal{P}_{\text{full}}$,
2) $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \leq p((\mathcal{P}_{\text{full}})_c)$.

Proof: (only if) As noted above, if $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection, then $\mathcal{C}' = \mathcal{C} + \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ with respect to $(\mathcal{P}_{\text{full}})_c$. We note that $\mathcal{C} \subseteq \mathcal{C}'$. Therefore if $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ and $C' = \begin{pmatrix} C'_1 & C'_2 \end{pmatrix}$ are minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then there exists a polynomial matrix F such that $C' = FC$. As C and C' have full row rank F also has full row rank. If c_2 is free in $\mathcal{C}$ then it is also free in $\mathcal{C}'$ (since if C_1 has full row rank then $C'_1 = FC_1$ will also have full row rank). As $\mathcal{C}'$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ through full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$, from theorem 11 it directly follows that

$$p((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \leq p((\mathcal{P}_{\text{full}})_c).$$

(if) Using theorem 11 and proposition 15 condition 1 and 2 together implies that there exists a controller $\mathcal{C}$ in which c_2 is free and regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ through full interconnection. From [6] and [7] the same $\mathcal{C}$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$). ■

Let k_1 and k_2 be the size of c_1 and c_2 respectively. Condition 2 of the above theorem requires $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \leq p((\mathcal{P}_{\text{full}})_c)$, equivalently $k_2 - m((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \leq (k_1 + k_2) - m((\mathcal{P}_{\text{full}})_c)$, therefore $m((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \geq m((\mathcal{P}_{\text{full}})_c) - k_1$. In other words for $\mathcal{K}$ to be regularly implementable through partial interconnection by a controller in which c_2 is free, the input cardinality of the projected behavior $(\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}$ should not be less than the difference between the input cardinality of $(\mathcal{P}_{\text{full}})_c$ and k_1.

Finally we prove a theorem, which gives necessary and sufficient conditions on $\mathcal{K}$ to be regularly implementable through partial interconnection by a controller $\mathcal{C}$ in which c_2 is *maximally* free.

Theorem 17: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ with system variable (w, c). Let $\mathcal{K} \in \mathcal{L}^q$. Partition $c = (c_1, c_2)$ with c_1 size k_1 and c_2 size k_2. Consider the following three conditions:

1) $\mathcal{K}$ is regularly implementable through c with respect to $\mathcal{P}_{\text{full}}$,
2) $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_{c_2}) \leq p((\mathcal{P}_{\text{full}})_c)$,
3) $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_c) = k_1 + p((\mathcal{P}_{\text{full}})_c)$, where k_1 is size of c_1.

If 1, 2 and 3 hold then $\mathcal{K}$ is regularly implementable by means of a controller in which c_2 is maximally free. If $\{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ is autonomous, then 1, 2, and 3 are also necessary for the existence of a controller $\mathcal{C}$ in which c_2 is maximally free that implements $\mathcal{K}$.

Before proving the theorem we will establish a result which will be useful in the proof.

Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ with plant variable (w, c), let $\mathcal{C}$ regularly implement $\mathcal{K} \in \mathcal{L}^q$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$). Define $\mathcal{N}_c(\mathcal{P}_{\text{full}}) = \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ and $\mathcal{C}' = \mathcal{C} + \mathcal{N}_c(\mathcal{P}_{\text{full}})$. Then we have the following lemma:

Lemma 18: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+k}$ with system variable (w, c). Let $\mathcal{C}, \mathcal{C}' \in \mathcal{L}^k$ regularly implement $\mathcal{K} \in \mathcal{L}^q$ through c with respect to $\mathcal{P}_{\text{full}}$, and $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ be autonomous. If C and C' are minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively, then $C' = FC$ where F is a square nonsingular polynomial matrix.

Proof: It is evident that $\mathcal{C} \subseteq \mathcal{C}'$, therefore there exists a polynomial matrix F such that $C' = FC$. As C, C' have full row rank F also has full row rank.

Let $\begin{pmatrix} R_1 & R_2 \end{pmatrix}$ give a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Then $\mathcal{N}_c(\mathcal{P}_{\text{full}}) = \ker(R_2)$, and is autonomous if and only if R_2 has full column rank.

As both $\mathcal{C}$ and $\mathcal{C}'$ regularly implement $\mathcal{K}$ by partial interconnection with respect to $\mathcal{P}_{\text{full}}$ we have

$$p(\mathcal{K}) = p((\mathcal{K}_{\text{full}}(\mathcal{C}))_w) = p((\mathcal{K}_{\text{full}}(\mathcal{C}'))_w). \tag{3}$$

Also from [9], $p((\mathcal{K}_{\text{full}}(\mathcal{C}))_w) = \operatorname{rank}\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix} - \operatorname{rank}\begin{pmatrix} R_2 \\ C \end{pmatrix}$, which is equivalent to

$$p((\mathcal{K}_{\text{full}}(\mathcal{C}))_w) = \operatorname{rank}\begin{pmatrix} R_1 & R_2 \end{pmatrix} + \operatorname{rank}(C) - \operatorname{rank}(R_2). \tag{4}$$

Similarly,

$$p((\mathcal{K}_{\text{full}}(\mathcal{C}'))_w) = \operatorname{rank}\begin{pmatrix} R_1 & R_2 \end{pmatrix} + \operatorname{rank}(C') - \operatorname{rank}(R_2). \tag{5}$$

Substituting equations 4 and 5 in equation 3, we get $\operatorname{rank}(C) = \operatorname{rank}(C')$. Which implies $\operatorname{rowdim}(F) = \operatorname{coldim}(F)$. ■

Using the above lemma we now prove theorem 17.

Proof of theorem 17: (only if) As the set of controllers in this theorem is a subset of the set of controllers treated in theorem 16, condition 1 and 2 directly follows from theorem 16.

From [6] and [7] if $\mathcal{C}$ regularly implements $\mathcal{K}$ then $\mathcal{C}' = \mathcal{C} + \{c \mid (0, c) \in \mathcal{P}_{\text{full}}\}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ with respect to $(\mathcal{P}_{\text{full}})_c$. Let $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ and $C' = \begin{pmatrix} C'_1 & C'_2 \end{pmatrix}$ be minimal kernel representations of $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then from lemma 18 there exists a polynomial matrix F which is square and nonsingular such that $\begin{pmatrix} C'_1 & C'_2 \end{pmatrix} = \begin{pmatrix} FC_1 & FC_2 \end{pmatrix}$. If c_2 is maximally free in $\mathcal{C}$ then it is also maximally free in $\mathcal{C}'$ (since if C_1 is square and nonsingular then the same will hold for FC_1).

As $\mathcal{C}'$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ by full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$, from theorem 14 it directly follows that $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_c) = k_1 + p((\mathcal{P}_{\text{full}})_c)$.

(if) Let $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ and $\begin{pmatrix} P_1 & P_2 \end{pmatrix}$ give minimal kernel representations of $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ and $(\mathcal{P}_{\text{full}})_c$ respectively. From condition 1, using proposition 15, there exists a matrix F such that $\begin{pmatrix} P_1 & P_2 \end{pmatrix} = \begin{pmatrix} FL_1 & FL_2 \end{pmatrix}$ and such that $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose W such that $\begin{pmatrix} F \\ W \end{pmatrix}$ forms a unimodular matrix. Then a parameterization of all controllers that regularly implement $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ with respect to $(\mathcal{P}_{\text{full}})_c$ is given by polynomial $\begin{pmatrix} C_1 & C_2 \end{pmatrix} = \begin{pmatrix} GP_1 + UWL_1 & GP_2 + UWL_2 \end{pmatrix}$, where G is an arbitrary polynomial matrix, U is unimodular. From theorem 14, condition 2 and 3 imply that there exists a G such that $GP_1 + WL_1$ has full row rank and is square.

Define $\mathcal{C}' = \ker\begin{pmatrix} GP_1 + WL_1 & GP_2 + WL_2 \end{pmatrix}$. Then $\mathcal{C}'$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ through full interconnection with respect to $(\mathcal{P}_{\text{full}})_c$ and in $\mathcal{C}'$ c_2 is maximally free. From [6] and [7], the same $\mathcal{C}'$ regularly implements $\mathcal{K}$ by partial interconnection (through c with respect to $\mathcal{P}_{\text{full}}$). ■

Condition 3 of the theorem requires $p((\mathcal{L}_{\text{full}}(\mathcal{K}))_c) = k_1 + p((\mathcal{P}_{\text{full}})_c)$, which is equivalent to $(k_1 + k_2) - m((\mathcal{L}_{\text{full}}(\mathcal{K}))_c) = k_1 + (k_1 + k_2) - m((\mathcal{P}_{\text{full}})_c)$, therefore $m((\mathcal{L}_{\text{full}}(\mathcal{K}))_c) = m((\mathcal{P}_{\text{full}})_c) - k_1$. In other words under the condition that $\mathcal{N}_c(\mathcal{P}_{\text{full}})$ is autonomous, for $\mathcal{K}$ to be regularly implementable through partial interconnection by a controller in which c_2 is maximally free the input cardinality of $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ should be equal to the difference between input cardinality of $(\mathcal{P}_{\text{full}})_c$ and k_1.

V. CONCLUSION

The problems of regular implementability by controllers in which some part of the control variables is free or maximally free are considered. For these problems necessary and sufficient conditions on the desired behaviors are obtained, both in the full interconnection and the partial interconnection case.

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